

# Vacuum polarization in thermal QED with an external magnetic field

J.Alexandre<sup>1</sup>

Department of Physics, National Technical University of Athens,  
Zografou Campus, 157 80 Athens, Greece

## Abstract

The one-loop vacuum polarization tensor is computed in QED with an external, constant, homogeneous magnetic field at finite temperature. The Schwinger proper-time formalism is used and the computations are done in Euclidian space. The well-known results are recovered when the temperature and/or the magnetic field are switched off and the effect of the magnetic field on the Debye screening is discussed.

## Introduction

The question of dynamical chiral symmetry breaking in thermal QED with an external magnetic field (magnetic catalysis) has been studied in the context of the electroweak transition [1] and also, with  $QED_3$  (in 2+1 dimensions), in the framework of effective descriptions of planar superconductors [2],[3]. Recent studies of the magnetic catalysis at zero temperature [4] showed that it is essential to take into account the momentum dependence of the fermion self-energy since the dynamical mass given by the constant self-energy approximation proved to be too small, by several orders of magnitude in the case of QED. These studies have been made with the analysis of the gap equation provided by the Schwinger-Dyson equation, where the photon propagator was truncated at the one-loop level. The polarization tensor in the presence of an external magnetic field was used in its lowest Landau level approximation, as was done in [5]. The study of the magnetic catalysis at finite temperature taking into account the momentum dependence of the fermion self-energy has been done in  $QED_3$  [3] but not in QED for which only the constant self-energy approximation has been done [6], [7]. With a study including the momentum dependence, we can still expect the critical temperature for the magnetic catalysis to be of the order of the dynamical mass found at zero temperature [6], but where the latter is given by a momentum-dependent analysis as was made in [4]. As a first step in this direction, we compute here the one-loop polarization tensor in finite temperature QED in the presence of a external, constant, homogeneous magnetic field.

The computation will be done in Euclidian space, using the proper-time formalism introduced by Schwinger [8] which takes into account the complete interaction between the fermion

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<sup>1</sup>jalex@central.ntua.gr

and the external, classical field. The same computation has been done at zero temperature in the paper [9] which will be often cited in the present article and the generalization to any external constant field is done in [10], using the 'string-inspired' technique. We note that the derivation of the Heisenberg-Euler lagrangian has been done at finite temperature with the same formalism [11], as well as the generalization to any external constant electromagnetic field [12].

Section 1 will introduce the notations and recall the characteristics of fermions in an external magnetic field. Section 2 will be devoted to the computation of the 44-component of the polarization tensor: this presentation is chosen for the sake of clarity since the external environment strongly breaks the symmetry between the Lorentz indices such that the computation is not straightforward. The technical details of the method will be explained and we will recover the well-known results in the limit where the temperature and/or the magnetic field go to zero. The other components will be computed in section 3 where the transversality of the polarization tensor will be checked. The section 4 will give the strong field approximation of the 44-component of the polarization tensor, consistent with the lowest Landau level approximation. Finally, the conclusion will show the Debye screening obtained through these computations.

## 1 Fermions in a constant magnetic field

To fix our notations we shortly review here the characteristics of fermions in a external, constant, homogeneous magnetic field at zero temperature.

The model we are going to consider is described by the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}D_{\mu}\gamma^{\mu}\Psi - m\bar{\Psi}\Psi, \quad (1)$$

where  $D_{\mu} = \partial_{\mu} + ieA_{\mu} + ieA_{\mu}^{ext}$ ,  $A_{\mu}$  is the abelian quantum gauge field,  $F_{\mu\nu}$  its corresponding field strength, and  $A_{\mu}^{ext}$  describes the external magnetic field. We recall the usual definition  $e^2 \equiv 4\pi\alpha$ .

We will choose the symmetric gauge for the external field ( $\vec{B}$  is in the direction 3)

$$A_0^{ext}(x) = 0, \quad A_1^{ext}(x) = -\frac{B}{2}x_2, \quad A_2^{ext}(x) = +\frac{B}{2}x_1, \quad A_3^{ext}(x) = 0 \quad (2)$$

for which we know from the work by Schwinger [8] that the fermion propagator is given by:

$$S(x, y) = e^{ie x^{\mu} A_{\mu}^{ext}(y)} \tilde{S}(x - y), \quad (3)$$

where the translational invariant propagator  $\tilde{S}$  has the following Fourier transform in the proper-time formalism:

$$\begin{aligned} \tilde{S}(p) &= \int_0^{\infty} ds e^{is(p_0^2 - p_3^2 - p_{\perp}^2 \frac{\tan(|eB|s)}{|eB|s} - m^2)} \\ &\times \left[ (p^0 \gamma^0 - p^3 \gamma^3 + m)(1 + \gamma^1 \gamma^2 \tan(|eB|s)) - p^{\perp} \gamma^{\perp} (1 + \tan^2(|eB|s)) \right] \end{aligned} \quad (4)$$

where  $p^{\perp} = (p^1, p^2)$  is the transverse momentum and the same notation holds for the gamma matrices.

Let us now turn to the finite temperature case. We will note the fermionic Matsubara modes  $\hat{\omega}_l = (2l + 1)\pi T$  and the bosonic ones  $\omega_n = 2n\pi T$ . The translational invariant part of the bare fermion propagator reads in Euclidian space ( $p_0 \rightarrow i\hat{\omega}_l$ ) and with the substitution  $s \rightarrow -is$ :

$$\begin{aligned} \tilde{S}_l(\vec{p}) &= -i \int_0^\infty ds e^{-s(\hat{\omega}_l^2 + p_3^2 + p_\perp^2 \frac{\tanh(|eB|s)}{|eB|s} + m^2)} \\ &\times \left[ (-\hat{\omega}_l \gamma^4 - p^3 \gamma^3 + m)(1 - i\gamma^1 \gamma^2 \tanh(|eB|s)) - p^\perp \gamma^\perp (1 - \tanh^2(|eB|s)) \right] \end{aligned} \quad (5)$$

where the Euclidian gamma matrices satisfy the anticommutation relation  $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$ , with  $\mu, \nu = 1, 2, 3, 4$  and  $\vec{p} = (p_\perp, p_3)$ .

Finally, the one-loop polarization tensor is

$$\Pi_n^{\mu\nu}(\vec{k}) = -4\pi\alpha T \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{l=-\infty}^{\infty} \text{tr} \left\{ \gamma^\mu \tilde{S}_l(\vec{p}) \gamma^\nu \tilde{S}_{l-n}(\vec{p} - \vec{k}) \right\} + Q^{\mu\nu}(k) \quad (6)$$

where  $Q^{\mu\nu}$ , usually called the 'contact term', cancels the ultraviolet divergences and therefore does not depend on the temperature or on the magnetic field since these give finite effects. The addition of this contact term is equivalent to the addition of the counterterm  $(1 - Z_3)F^{\mu\nu}F_{\mu\nu}/4$  in the original Lagrangian and the usual ultraviolet divergences appear in the proper-time formalism as singularities in  $s = 0$ , as will be seen in the next section. With this proper-time method, a cut-off  $0 < \varepsilon < s$  provides a gauge invariant regularization which will be used in what follows. The limit  $\varepsilon \rightarrow 0$  will be taken after computing the contact term  $Q^{\mu\nu}$ .

We note that the  $A_\mu^{ext}$ -dependent phase of the fermion propagator does not contribute to the polarization tensor since in coordinate space

$$\exp \left\{ ie \left( x^\mu A_\mu^{ext}(y) + y^\mu A_\mu^{ext}(x) \right) \right\} = 1 \quad (7)$$

with the specific choice of gauge (2). If we choose another potential  $A_\mu^{ext}$ , it is shown in [12] that the change of gauge is equivalent to the introduction of a chemical potential.

## 2 44-component

With the expression (5) of the fermion propagator, we obtain for the 44-component of the polarization tensor after the integration over  $\vec{p}$

$$\begin{aligned} \Pi_n^{44}(\vec{k}) &= \frac{-2\alpha T}{\sqrt{\pi}} |eB| \int \frac{ds d\sigma}{\sqrt{s + \sigma} (\tanh(|eB|s) + \tanh(|eB|\sigma))} \\ &\times \sum_{l=-\infty}^{\infty} e^{-\frac{k_\perp^2}{|eB|} \frac{\tanh(|eB|s) \tanh(|eB|\sigma)}{\tanh(|eB|s) + \tanh(|eB|\sigma)} - [(s + \sigma)(\hat{\omega}_l^2 + m^2) + s\omega_n(\omega_n - 2\hat{\omega}_l) + \frac{s\sigma}{s + \sigma} k_3^2]} \\ &\times \left[ k_\perp^2 \frac{\tanh(|eB|s) \tanh(|eB|\sigma)}{(\tanh(|eB|s) + \tanh(|eB|\sigma))^2} (1 - \tanh(|eB|s))(1 - \tanh(|eB|\sigma)) \right. \\ &\quad \left. - |eB| \frac{(1 - \tanh(|eB|s))(1 - \tanh(|eB|\sigma))}{\tanh(|eB|s) + \tanh(|eB|\sigma)} \right. \\ &\quad \left. + \left( \hat{\omega}_l(\hat{\omega}_l - \omega_n) - m^2 + \frac{s\sigma}{(s + \sigma)^2} k_3^2 - \frac{1}{2(s + \sigma)} \right) (1 + \tanh(|eB|s) \tanh(|eB|\sigma)) \right] + Q^{44}(k) \end{aligned} \quad (8)$$

In finite temperature computations, one usually first does the summation over Matsubara modes and then the integration over momenta. In this formalism, what is important as we will see below is to do the summation over Matsubara modes before the integration over the proper-time parameters, when the cut-off is removed (i.e.  $\varepsilon \rightarrow 0$ ). As in [9], we make the change of variable  $s = u(1 - v)/2$  and  $\sigma = u(1 + v)/2$  to obtain

$$\begin{aligned} \Pi_n^{44}(\vec{k}) &= \frac{-\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-\frac{k_{\perp}^2}{|eB|} \frac{\cosh \bar{u} - \cosh \bar{u}v}{2 \sinh \bar{u}} - u[\hat{\omega}_l^2 + m^2 + (1-v)\omega_n(\omega_n/2 - \hat{\omega}_l) + \frac{1-v^2}{4}k_3^2]} \\ &\times \left[ \left( \hat{\omega}_l(\hat{\omega}_l - \omega_n) + \frac{1-v^2}{4}k_3^2 - \frac{1}{2u} - m^2 \right) \coth \bar{u} - \frac{|eB|}{\sinh^2 \bar{u}} + k_{\perp}^2 \frac{\cosh \bar{u} - \cosh \bar{u}v}{2 \sinh^3 \bar{u}} \right] + Q^{44}(k) \end{aligned}$$

where  $\bar{u} = |eB|u$ . We make the integration by parts over  $u$  (we note  $\phi(u)$  the exponent)

$$|eB| \int_{\varepsilon}^{\infty} du e^{-\phi(u)} \frac{\sqrt{u}}{\sinh^2 \bar{u}} \longrightarrow \int_{\varepsilon}^{\infty} du e^{-\phi(u)} \sqrt{u} \coth \bar{u} \left( \frac{1}{2u} - \frac{d\phi(u)}{du} \right) \quad (9)$$

where we disregard the surface term [9]. We then obtain the final expression

$$\begin{aligned} \Pi_n^{44}(\vec{k}) &= \frac{-\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-\frac{k_{\perp}^2}{|eB|} \frac{\cosh \bar{u} - \cosh \bar{u}v}{2 \sinh \bar{u}} - u[m^2 + W_l^2 + \frac{1-v^2}{4}(\omega_n^2 + k_3^2)]} \\ &\times \left[ \frac{k_{\perp}^2}{2} \frac{\cosh \bar{u}v - v \coth \bar{u} \sinh \bar{u}v}{\sinh \bar{u}} - \coth \bar{u} \left( \frac{1}{u} - 2W_l^2 + v\omega_n W_l - \frac{1-v^2}{2}k_3^2 \right) \right] + Q^{44}(k) \end{aligned} \quad (10)$$

where  $W_l = \hat{\omega}_l - \frac{(1-v)}{2}\omega_n$ . We note for the purpose of consistency that the integrand in (10) is an even function of the parameter  $v$  since  $W_l(-v) = -W_{n-l-1}(v)$  and therefore  $\sum_l e^{-uW_l^2}$  is even in  $v$ , which ensures the symmetry between the proper-times  $s$  and  $\sigma$ . Thus it is important to perform the summation over Matsubara modes before doing the integrations over the proper-time parameters. Another reason to do the summation over Matsubara modes first is to avoid artificial divergences in the temperature-dependent part of the polarization tensor (which should be finite), as will be seen at the end of this section.

Let us now determine the contact term  $Q^{44}(k)$ . Since it does not depend on the temperature or the magnetic field, it will be determined after taking the limit  $T \rightarrow 0$  and  $|eB| \rightarrow 0$  of (10). If we set  $T = 0$  in (10), we recover the zero-temperature results given in [9] since the substitutions  $W_l \rightarrow p_4$  and  $T \sum_l \rightarrow (2\pi)^{-1} \int dp_4$  lead to

$$\begin{aligned} \lim_{T \rightarrow 0} T \sum_{l=-\infty}^{\infty} e^{-uW_l^2} &= \frac{1}{2\sqrt{\pi u}} \\ \lim_{T \rightarrow 0} T \sum_{l=-\infty}^{\infty} \left( \frac{1}{u} - 2W_l^2 \right) e^{-uW_l^2} &= 0 \\ \lim_{T \rightarrow 0} T \sum_{l=-\infty}^{\infty} v\omega_n W_l e^{-uW_l^2} &= 0 \end{aligned} \quad (11)$$

and therefore ( $\omega_n \rightarrow k_4$ )

$$\begin{aligned} \lim_{T \rightarrow 0} \Pi_n^{44}(\vec{k}) &= \frac{-\alpha |eB|}{4\pi} \int_{\varepsilon}^{\infty} du \int_{-1}^1 dv e^{-\frac{k_{\perp}^2}{|eB|} \frac{\cosh \bar{u} - \cosh \bar{u} v}{2 \sinh \bar{u}} - u[m^2 + \frac{1-v^2}{4}(k_4^2 + k_3^2)]} \\ &\times \left[ k_{\perp}^2 \frac{\cosh \bar{u} v - v \coth \bar{u} \sinh \bar{u} v}{\sinh \bar{u}} + k_3^2 (1-v^2) \coth \bar{u} \right] + Q^{44}(k) \end{aligned} \quad (12)$$

However it is important to keep a non zero fermion mass if we wish to recover this infrared limit: we will see in the conclusion that if we set  $m = 0$ ,  $\Pi_0^{44}(0)$  reaches a non-zero value in the limit  $T \rightarrow 0$ . Thus we can commute the limit  $T \rightarrow 0$  and the integration over the proper-time  $u$  to find a consistent zero temperature result only if  $m \neq 0$ , at least as long as  $|eB| > 0$ . This condition is consistent since the magnetic catalysis generates a dynamical mass when the temperature is lower than the critical one, for any value of the gauge coupling, which has been proven at least for strong magnetic fields [5]. Therefore if the fermion should be massless, the use of the dynamically generated mass  $m_{dyn}$  instead of  $m = 0$  in (4), corresponding to a resummation of graphs for the fermion propagator, would lead us back to the massive case. Thus we will consider  $m \neq 0$  in what follows.

If we now wish to take the limit of zero magnetic field of (12), we take  $\bar{u} \rightarrow 0$  and obtain

$$\lim_{T \rightarrow 0, |eB| \rightarrow 0} \Pi_n^{44}(\vec{k}) = \frac{-\alpha}{4\pi} \int_{\varepsilon}^{\infty} \frac{du}{u} \int_{-1}^1 dv e^{-u[m^2 + \frac{1-v^2}{4}k^2]} (1-v^2) \vec{k}^2 + Q^{44}(k) \quad (13)$$

where  $k^2 = k_4^2 + \vec{k}^2$ . Then if we take the contact term

$$Q^{44}(k) = \frac{\alpha}{4\pi} \int_{\varepsilon}^{\infty} \frac{du}{u} \int_{-1}^1 dv e^{-um^2} (1-v^2) \vec{k}^2 \quad (14)$$

we obtain finally when  $\varepsilon \rightarrow 0$

$$\begin{aligned} \lim_{T \rightarrow 0, |eB| \rightarrow 0} \Pi_n^{44}(\vec{k}) &= \frac{-\alpha}{4\pi} \int_0^{\infty} \frac{du}{u} \int_{-1}^1 dv (1-v^2) \left( e^{-u[m^2 + \frac{1-v^2}{4}k^2]} - e^{-um^2} \right) \vec{k}^2 \\ &= \frac{\alpha}{4\pi} \int_{-1}^1 dv (1-v^2) \ln \left( 1 + \frac{1-v^2}{4m^2} k^2 \right) \vec{k}^2 \end{aligned} \quad (15)$$

which is a result obtained by standard methods [13] with the Feynman parameter  $z = (1+v)/2$ .

To finish the comparison with results already established, let us take the zero magnetic field limit of (10). The limit  $\bar{u} \rightarrow 0$  leads to

$$\begin{aligned} \lim_{|eB| \rightarrow 0} \Pi_n^{44}(\vec{k}) &= \frac{-\alpha T}{\sqrt{\pi}} \int_{\varepsilon}^{\infty} \frac{du}{\sqrt{u}} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-u[m^2 + W_l^2 + \frac{1-v^2}{4}(\omega_n^2 + \vec{k}^2)]} \\ &\times \left[ \frac{\vec{k}^2}{2} (1-v^2) - \frac{1}{u} + 2W_l^2 - v\omega_n W_l \right] + Q^{44}(k) \end{aligned} \quad (16)$$

For  $|eB| = 0$ , we can take a massless fermion ( $m = 0$ ) since there is no magnetic catalysis and the Debye screening is then given by

$$M_{|eB|=0, m=0}^2(T) = - \lim_{\vec{k}^2 \rightarrow 0} \Pi_0^{44}(\vec{k})|_{|eB|=0} = c \alpha T^2 \quad (17)$$

with ( $\varepsilon \rightarrow 0$ )

$$c = 2\sqrt{\pi} \int_0^\infty \frac{du}{\sqrt{u}} \sum_{l=-\infty}^\infty e^{-u(2l+1)^2} \left[ 2(2l+1)^2 - \frac{1}{u} \right] \quad (18)$$

To compute  $c$ , we use the Poisson resummation [11]:

$$\sum_{l=-\infty}^\infty e^{-a(l-z)^2} = \left( \frac{\pi}{a} \right)^{1/2} \sum_{l=-\infty}^\infty e^{-\frac{\pi^2 l^2}{a} - 2i\pi z l} \quad (19)$$

which shows that it is essential to perform the summation over Matsubara modes before doing the integration over the proper-time  $u$  to avoid the singularity  $\int du u^{-3/2}$  in (18) since we obtain

$$\sum_{l=-\infty}^\infty e^{-u(2l+1)^2} \left[ 2(2l+1)^2 - \frac{1}{u} \right] = \frac{\pi^{5/2}}{2u^{5/2}} \sum_{l \geq 1} (-1)^{l+1} l^2 e^{-\frac{\pi^2 l^2}{4u}} \quad (20)$$

such that

$$c = \pi^3 \int_0^\infty \frac{du}{u^3} \sum_{l \geq 1} (-1)^{l+1} l^2 e^{-\frac{l^2 \pi^2}{4u}} = \frac{16}{\pi} \int_0^\infty dx x e^{-x} \sum_{l \geq 1} \frac{(-1)^{l+1}}{l^2} = \frac{4\pi}{3} \quad (21)$$

which gives the well known result for the one-loop Debye screening with massless fermions [14] for which higher order corrections can be found in [15]. We note that we can commute the integration over the proper-time and the summation over the Matsubara modes after doing the Poisson resummation.

Using again the Poisson resummation (19), we can give another form of  $\Pi_n^{44}(\vec{k})$  which splits the temperature independent part from the temperature dependent one. A straightforward computation leads to

$$\Pi_n^{44}(\vec{k}) = \Pi_n^0(\vec{k}) + \Pi_n^T(\vec{k}) \quad (22)$$

where  $\Pi_n^0(\vec{k})$  is the zero temperature part (12) (with  $k_4 \rightarrow \omega_n$ ) and  $\Pi_n^T(\vec{k})$  the temperature dependent part

$$\begin{aligned} \Pi_n^T(\vec{k}) = & \frac{-\alpha}{2\pi} |eB| \int_0^\infty du \int_{-1}^1 dv e^{-\frac{k_\perp^2}{|eB|} \frac{\cosh \bar{u} - \cosh \bar{u} v}{2 \sinh \bar{u}} - u[m^2 + \frac{1-v^2}{4}(\omega_n^2 + k_3^2)]} \\ & \times \sum_{l \geq 1} (-1)^l e^{-\frac{l^2}{4uT^2}} \left[ \left( k_\perp^2 \frac{\cosh \bar{u} v - v \coth \bar{u} \sinh \bar{u} v}{\sinh \bar{u}} + k_3^2 (1-v^2) \coth \bar{u} \right) \cos \pi n l (1-v) \right. \\ & \left. - \frac{\coth \bar{u}}{u} \left( \frac{l^2}{uT^2} \cos \pi n l (1-v) - 2\pi v n l \sin \pi n l (1-v) \right) \right] \end{aligned} \quad (23)$$

where we took  $\varepsilon \rightarrow 0$  since the temperature dependent part is finite. We see again that after this Poisson resummation every term of the Matsubara series gives a finite integration over the proper-time  $u$ .

### 3 Other components and transversality

We now compute the other components in a similar way and will give only the important steps. We first give the diagonal components of the polarization tensor which all need integrations by parts to lead to the good limit when  $T \rightarrow 0$ . We set

$$\phi_l(u, v) = \frac{k_\perp^2}{2|eB|} \frac{\cosh \bar{u} - \cosh \bar{u}v}{\sinh \bar{u}} + u \left[ m^2 + W_l^2 + \frac{1-v^2}{4}(\omega_n^2 + k_3^2) \right] \quad (24)$$

Let us start with  $\Pi_n^{33}(\vec{k})$ : the same steps as the ones used for the computation of  $\Pi_n^{44}(\vec{k})$  and the same integration by parts lead to

$$\begin{aligned} \Pi_n^{33}(\vec{k}) &= \frac{-\alpha T}{\sqrt{\pi}} |eB| \int_\varepsilon^\infty du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^\infty e^{-\phi_l(u,v)} \\ &\times \left[ v \omega_n W_l \coth \bar{u} + \frac{k_\perp^2}{2} \frac{\cosh \bar{u}v - v \coth \bar{u} \sinh \bar{u}v}{\sinh \bar{u}} + \omega_n^2 \frac{1-v^2}{2} \coth \bar{u} \right] + Q^{33}(k) \end{aligned} \quad (25)$$

The computation of  $\Pi_n^{ii}(\vec{k})$ ,  $i = 1, 2$  (without summation over  $i$ ) is slightly different. After the integration over the loop momentum  $\vec{p}$ , the change of variable  $s = u(1-v)/2$  and  $\sigma = u(1+v)/2$  leads to

$$\begin{aligned} \Pi_n^{ii}(\vec{k}) &= \frac{-\alpha T}{\sqrt{\pi}} |eB| \int_\varepsilon^\infty du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^\infty e^{-\phi_l(u,v)} \left[ \frac{\cosh \bar{u}v}{\sinh \bar{u}} \left( v \omega_n W_l - \frac{1}{2u} - W_l^2 - m^2 \right) \right. \\ &\quad \left. - \frac{2k_i^2 - k_\perp^2}{2} \frac{\cosh \bar{u} - \cosh \bar{u}v}{\sinh^3 \bar{u}} + (\omega_n^2 + k_3^2) \frac{1-v^2}{4} \frac{\cosh \bar{u}v}{\sinh \bar{u}} \right] + Q^{ii}(k) \end{aligned} \quad (26)$$

Then we make the integration by parts over  $u$

$$\begin{aligned} \int_\varepsilon^\infty du e^{-\phi_l(u,v)} m^2 \sqrt{u} \frac{\cosh \bar{u}v}{\sinh \bar{u}} &\longrightarrow \\ \int_\varepsilon^\infty du e^{-\phi_l(u,v)} \sqrt{u} \frac{\cosh \bar{u}v}{\sinh \bar{u}} &\left[ \frac{1}{2u} + |eB| (v \tanh \bar{u}v - \coth \bar{u}) - \frac{d}{du} (\phi_l(u, v) - u m^2) \right] \end{aligned} \quad (27)$$

followed by the intergration by parts over  $v$

$$\begin{aligned} \int_{-1}^1 dv e^{-\phi_l(u,v)} \frac{\cosh \bar{u}v}{\sinh \bar{u}} &\left[ \frac{1}{u} + |eB| (v \tanh \bar{u}v - \coth \bar{u}) \right] \longrightarrow \\ \int_{-1}^1 dv e^{-\phi_l(u,v)} \frac{v \cosh \bar{u}v - \coth \bar{u} \sinh \bar{u}v}{\sinh \bar{u}} &\left[ \omega_n W_l - \frac{k_\perp^2}{2} \frac{\sinh \bar{u}v}{\sinh \bar{u}} - \frac{v}{2} (\omega_n^2 + k_3^2) \right] \end{aligned} \quad (28)$$

where we again disregarded the surface terms. We finally obtain

$$\begin{aligned} \Pi_n^{ii}(\vec{k}) &= \frac{-\alpha T}{\sqrt{\pi}} |eB| \int_\varepsilon^\infty du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^\infty e^{-\phi_l(u,v)} \left[ \coth \bar{u} \frac{\sinh \bar{u}v}{\sinh \bar{u}} \omega_n W_l \right. \\ &+ (k_\perp^2 - k_i^2) \frac{\cosh \bar{u} - \cosh \bar{u}v}{\sinh^3 \bar{u}} + \frac{\omega_n^2 + k_3^2}{2} \frac{\cosh \bar{u}v - v \coth \bar{u} \sinh \bar{u}v}{\sinh \bar{u}} \left. \right] + Q^{ii}(k) \end{aligned} \quad (29)$$

Now let us go to the off-diagonal components of the polarization tensor. What differs from the diagonal components is that we do not make any integration by parts and we obtain directly the final results with the expected limit when  $T \rightarrow 0$ :

$$\begin{aligned}
\Pi_n^{34}(\vec{k}) &= \frac{\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-\phi_l(u,v)} k_3 \left[ v W_l + \frac{1-v^2}{2} \omega_n \right] \coth \bar{u} + Q^{34}(k) \\
\Pi_n^{12}(\vec{k}) &= \frac{\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-\phi_l(u,v)} k_1 k_2 \frac{\cosh \bar{u} - \cosh \bar{u} v}{\sinh^3 \bar{u}} + Q^{12}(k) \\
\Pi_n^{i4}(\vec{k}) &= \frac{\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-\phi_l(u,v)} \\
&\quad \times k_i \left[ W_l \coth \bar{u} \frac{\sinh \bar{u} v}{\sinh \bar{u}} + \frac{\omega_n \cosh \bar{u} v - v \coth \bar{u} \sinh \bar{u} v}{\sinh \bar{u}} \right] + Q^{i4}(k) \\
\Pi_n^{i3}(\vec{k}) &= \frac{\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} du \sqrt{u} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-\phi_l(u,v)} \frac{k_i k_3}{2} \frac{\cosh \bar{u} v - v \coth \bar{u} \sinh \bar{u} v}{\sinh \bar{u}} + Q^{i3}(k)
\end{aligned} \tag{30}$$

where  $i = 1, 2$ . It is easy to check that all the components of the polarization tensor give the results found in [9] when  $T \rightarrow 0$ . The contact terms are determined in the same way as  $Q^{44}(k)$  and can be summarised as

$$\begin{aligned}
Q^{\mu\nu}(k) &= \frac{\alpha}{4\pi} \int_{\varepsilon}^{\infty} \frac{du}{u} \int_{-1}^1 dv e^{-um^2} (1-v^2) (\delta^{\mu\nu} k^2 - k^{\mu} k^{\nu}) \\
&= \frac{\alpha}{3\pi} \int_{\varepsilon}^{\infty} \frac{du}{u} e^{-um^2} (\delta^{\mu\nu} k^2 - k^{\mu} k^{\nu})
\end{aligned} \tag{31}$$

as in [9].

It is important now to check the transversality of the polarization tensor, which is not obvious since  $\Pi^{\mu\nu}$  contains terms which are not explicitly proportional to any external momentum component. The contact term (31) is obviously transverse and with the expressions (10), (25), (29) and (30), we obtain

$$\begin{aligned}
k_{\nu} \Pi_n^{\nu\mu}(\vec{k}) &= \omega_n \Pi_n^{4\mu}(\vec{k}) + k_3 \Pi_n^{3\mu}(\vec{k}) + k_i \Pi_n^{i\mu}(\vec{k}) \\
&= \delta^{4\mu} \frac{\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} du \sqrt{u} \coth \bar{u} \int_{-1}^1 dv \sum_{l=-\infty}^{\infty} e^{-\phi_l(u,v)} \\
&\quad \times \left[ \frac{\omega_n}{u} + W_l \left( v(\omega_n^2 + k_3^2) + \frac{\sinh \bar{u} v}{\sinh \bar{u}} k_{\perp}^2 - 2\omega_n W_l \right) \right] \\
&= 2\delta^{4\mu} \frac{\alpha T}{\sqrt{\pi}} |eB| \int_{\varepsilon}^{\infty} \frac{du}{\sqrt{u}} \coth \bar{u} \int_{-1}^1 dv \frac{d}{dv} \left( \sum_{l=-\infty}^{\infty} W_l e^{-\phi_l(u,v)} \right) \\
&= \text{surface term}
\end{aligned} \tag{32}$$

so that the polarization tensor is transverse, since the above sum is zero up to surface terms which are normally omitted in this formalism.



## 4 Strong field approximation

We give here the strong field approximation of the 44-component of the polarization tensor that can be used in a strong field study of the magnetic catalysis.

The strong field asymptotic form of the fermion propagator (5) can be found by taking the limit  $|eB| \rightarrow \infty$  in the integrand, neglecting the shrinking region of integration where the product  $s|eB|$  goes to zero [16]. We obtain then

$$\begin{aligned}\tilde{S}_l(\vec{p}) &\simeq -i \int_0^\infty ds e^{-(\hat{\omega}_l^2 + p_3^2 + \frac{p_\perp^2}{|eB|s} + m^2)} \left( -\hat{\omega}_l \gamma^4 - p^3 \gamma^3 + m \right) (1 - i\gamma^1 \gamma^2) \\ &= i e^{-\frac{p_\perp^2}{|eB|}} \frac{\hat{\omega}_l \gamma^4 + p^3 \gamma^3 - m}{\hat{\omega}_l^2 + p_3^2 + m^2} (1 - i\gamma^1 \gamma^2)\end{aligned}\quad (33)$$

which is the well-known lowest Landau level approximation for the fermion propagator [5] that we can obtain by truncating the expansion of the propagator over the Landau levels to the dominant term [17]. We will take the same limit  $|eB| \rightarrow \infty$  in the expressions (12) and (23) to find the asymptotic form of  $\Pi^{44}$ . We note that the polarization tensor does not contain divergences in the limit  $|eB| \rightarrow \infty$ , the magnetic field playing the role of a cut-off for the momenta. This is due to the exponential decrease of the transverse degrees of freedom as can be seen in (33). Therefore we will not consider any contact term here.

The temperature-independent part (12) reads in the strong field limit ( $\bar{u} \rightarrow \infty$ )

$$\begin{aligned}\Pi_n^0(\vec{k}) &\simeq -\frac{\alpha|eB|}{4\pi} \int_0^\infty du \int_{-1}^1 dv e^{-\frac{k_\perp^2}{2|eB|} - u[m^2 + \frac{1-v^2}{4}(\omega_n^2 + k_3^2)]} \\ &\quad \times \left\{ k_\perp^2 \left[ (1-v)e^{-\bar{u}(1-v)} + (1+v)e^{-\bar{u}(1+v)} \right] + k_3^2(1-v^2) \right\} \\ &= -\frac{\alpha|eB|}{4\pi} e^{-\frac{k_\perp^2}{2|eB|}} \int_{-1}^1 dv \left[ \frac{(1-v^2)k_3^2}{m^2 + \frac{1-v^2}{4}(\omega_n^2 + k_3^2)} + \mathcal{O}\left(\frac{k_\perp^2}{|eB|}\right) \right]\end{aligned}\quad (34)$$

The integration over  $v$  of the dominant term leads then to the following expression that was already derived in [16] where the authors started the computation with the propagator (33):

$$\Pi_n^0(\vec{k}) \simeq -\frac{2\alpha}{\pi} |eB| \frac{k_3^2}{k_\parallel^2} e^{-\frac{k_\perp^2}{2|eB|}} \left[ 1 - \frac{2m^2}{\sqrt{k_\parallel^2(4m^2 + k_\parallel^2)}} \ln \left( \frac{\sqrt{4m^2 + k_\parallel^2} + \sqrt{k_\parallel^2}}{\sqrt{4m^2 + k_\parallel^2} - \sqrt{k_\parallel^2}} \right) \right] \quad (35)$$

where  $k_\parallel^2 = \omega_n^2 + k_3^2$ .

For the temperature-dependent part (23), the limit  $\bar{u} \rightarrow \infty$  followed by the change of variable  $u \rightarrow u/|eB|$  leads to the dominant term

$$\Pi_n^T(\vec{k}) \simeq -\frac{\alpha}{2\pi} \frac{|eB|^2}{T^2} e^{-\frac{k_\perp^2}{2|eB|}} \int_0^\infty \frac{du}{u^2} \int_{-1}^1 dv e^{-u\frac{\mu^2(v)}{4|eB|}} \sum_{l \geq 1} (-1)^{l+1} l^2 e^{-\frac{l^2|eB|}{4uT^2}} \cos \pi n l (1-v) \quad (36)$$

where  $\mu^2(v) = 4m^2 + (1-v^2)k_\parallel^2$ . We recognize here the Bessel function  $K_1$  since

$$\int_0^\infty \frac{du}{u^2} e^{-au - \frac{b}{u}} = \sqrt{\frac{a}{b}} \int_0^\infty du e^{-(u + \frac{1}{u})\sqrt{ab}} = 2\sqrt{\frac{a}{b}} K_1(2\sqrt{ab}) \quad (37)$$

where we defined

$$a = \frac{\mu^2(v)}{4|eB|} \quad \text{and} \quad b = \frac{l^2|eB|}{4T^2} \quad (38)$$

The temperature-dependent part can finally be written

$$\Pi_n^T(\vec{k}) \simeq -\frac{2\alpha}{\pi} |eB| e^{-\frac{k_\perp^2}{2|eB|}} \sum_{l \geq 1} (-1)^{l+1} \int_{-1}^1 dv \frac{l\mu(v)}{2T} K_1\left(\frac{l\mu(v)}{2T}\right) \cos \pi n l (1-v) \quad (39)$$

We will see in the conclusion that the previous sum and integrals can be evaluated for the computation of the Debye screening in the regime where  $m \ll T \ll \sqrt{|eB|}$ .

## Conclusion: Debye screening in a magnetic field

To conclude, we look in more details at the Debye screening obtained in this computation. From equation (23), we find for the Debye mass

$$M_{|eB|}^2(T) = -\lim_{\vec{k}^2 \rightarrow 0} \Pi_0^T(\vec{k}) = \frac{\alpha|eB|}{\pi T^2} \int_0^\infty \frac{du}{u^2} \coth \bar{u} e^{-um^2} \sum_{l \geq 1} (-1)^{l+1} l^2 e^{-\frac{l^2}{4uT^2}} \quad (40)$$

The zero-magnetic field limit is

$$M_{|eB|=0}^2(T) = \frac{\alpha}{\pi T^2} \int_0^\infty \frac{du}{u^3} e^{-um^2} \sum_{l \geq 1} (-1)^{l+1} l^2 e^{-\frac{l^2}{4uT^2}} \quad (41)$$

For given values of  $|eB|$  and  $m$ , in figure 1 we compare the ratios  $M_{|eB|}^2/|eB|$  and  $M_{|eB|=0}^2/|eB|$  as functions of  $T/\sqrt{|eB|}$ , such that all the dimensionful quantities are rescaled in units of the magnetic field ( $|eB| = 2$ ). For high temperatures the curves converge towards the result (17) (rescaled by  $|eB|$ ) since  $m \ll T$  and  $\sqrt{|eB|} \ll T$ , but for strong magnetic field  $T \ll \sqrt{|eB|}$ , a strong Debye screening is generated compared to the one without external field. As long as the temperature remains greater than the fermion mass, the Debye screening follows a plateau when the temperature decreases. We note that if we had  $m = 0$ , the limit of the Debye screening when  $T \rightarrow 0$  would be (after the change of variable  $u \rightarrow u/T^2$  in (40))

$$\begin{aligned} \lim_{T \rightarrow 0} M_{|eB|, m=0}^2(T) &= \frac{\alpha|eB|}{\pi} \int_0^\infty \frac{du}{u^2} \sum_{l \geq 1} (-1)^{l+1} l^2 e^{-\frac{l^2}{4u}} \\ &= \frac{4\alpha}{\pi} |eB| \int_0^\infty dx \sum_{l \geq 1} (-1)^{l+1} l e^{-xl} \\ &= \frac{4\alpha}{\pi} |eB| \lim_{x \rightarrow 0} \sum_{l \geq 1} (-1)^{l+1} e^{-xl} \end{aligned}$$

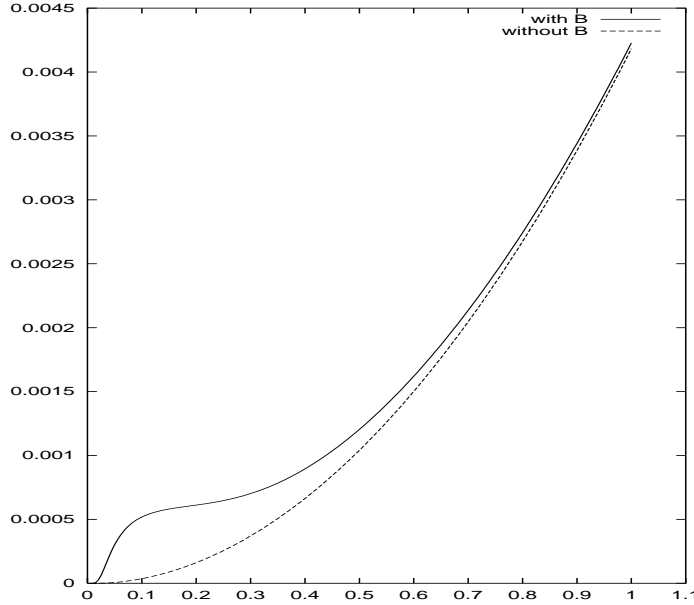


Figure 1:  $M^2_{|eB|}/|eB|$  and  $M^2_{|eB|=0}/|eB|$  versus  $T/\sqrt{|eB|}$  for  $\alpha = .001$  and  $m/\sqrt{|eB|} = .1$

$$\begin{aligned}
&= \frac{4\alpha}{\pi}|eB| \lim_{x \rightarrow 0} \left(1 - \frac{1}{1 + e^{-x}}\right) \\
&= \frac{2\alpha}{\pi}|eB|
\end{aligned} \tag{42}$$

But when  $T < m$  the fermion mass forces the screening to vanish with the temperature, so that the value (42) of the plateau is valid only if  $m \ll T \ll \sqrt{|eB|}$ .

We note that a more unexpected behaviour has been observed in  $QED_3$  at finite temperature in an external magnetic field [3]:  $M^2_{|eB|}$  first increases when the temperature decreases (in the region  $T \ll \sqrt{|eB|}$ ), reaches a maximum when  $T \simeq m$  and then decreases to 0 when  $T \rightarrow 0$ .

These behaviours of the Debye screening are a consequence of the dimensional reduction of the fermion dynamics in a strong magnetic field, as can be seen with the propagator (33).

To conclude, we note again the consistency between the necessity to have a massive fermion to obtain the good zero temperature limits (as long as  $|eB| > 0$ ) and the occurrence of the magnetic catalysis which generates dynamically this mass.

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